Basics of Turbulence I:

A Look at Homogeneous Systems

P.H. Diamond

UC San Diego and SWIP

1st Chengdu Theory Festival

Aug. 2018

Approach

- Highly Pedagogic

Turbulence

<u>scale</u>

Homogeneous Problems I
- Cascade
- Spectra
- Wave Interactions

…

<u>space</u>

Inhomogeneous Problems II
- Mixing length, profiles
- Pipe, wake flow
- 'Turbulence spreading'
- Avalanches

…

- Focus on simplest problems

Outline

- Basic Ideas
- K41 and Beyond
- Turbulence in Flatland – 2D Fluid Turbulence
- First Look at MHD Turbulence

Model

- Unless otherwise noted:

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}-\nu\nabla^2\vec{v}\right)=-\nabla P+\tilde{f}$$

$$\nabla\cdot\vec{v}=0$$

Random forcing
(usually large scale)

  - Finite domain, closed, periodic
  - $Re = v\cdot\nabla v\,/\nu\nabla^2 v \sim VL/\nu$; $Re \gg 1$

- Variants:
  - 2D, QG
  - Compressible flow
  - Pipe flow – inhomogeneity

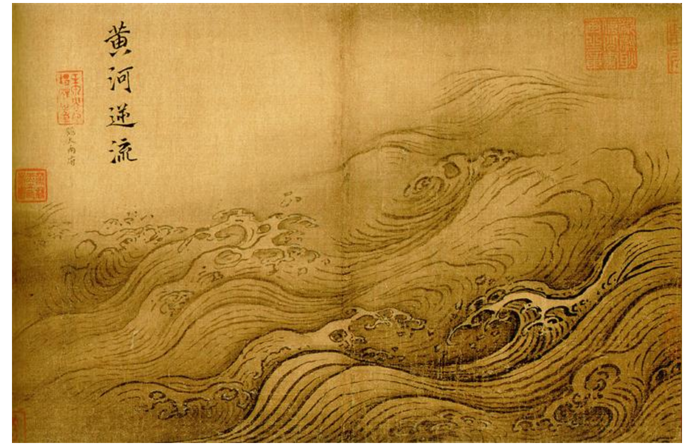
  ....

What is turbulence?

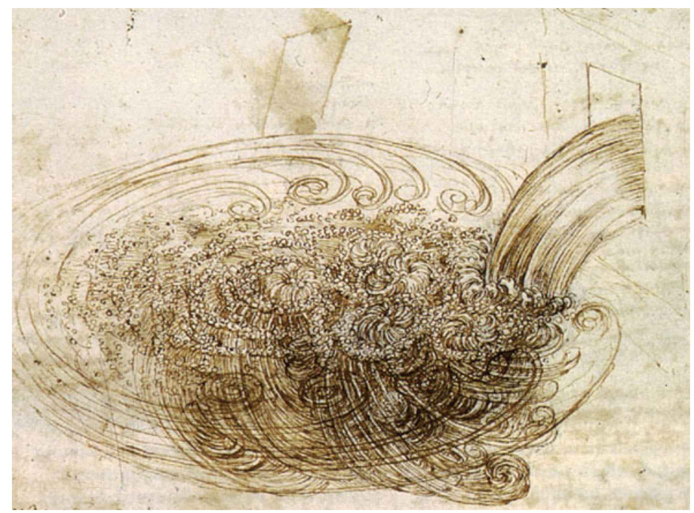
- Spatio-temporal “disorder”
- Broad range of space-time scales
- Power transfer / flux thru broad range of scales *
- Energy dissipation and irreversibility as $Re \to \infty$ *

And:

- Decay of large scales
- Irreversible mixing
- Intermittency / burstiness



Ma Yuan

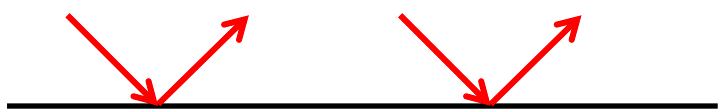


Leonardo

What is difference between turbulence and noise/equilibrium fluctuations?

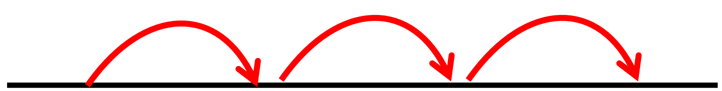
- Power transfer dominant
- Irreversibility for $\nu \to 0$
- Noisey thermal equilibrium: (ala' Test Particle Model)

  Emission <-> absorption balance, <u>locally</u>



Fluctuation-Dissipation Theorem applies

- Turbulence:



~ cascade

Flux ~ emission – absorption

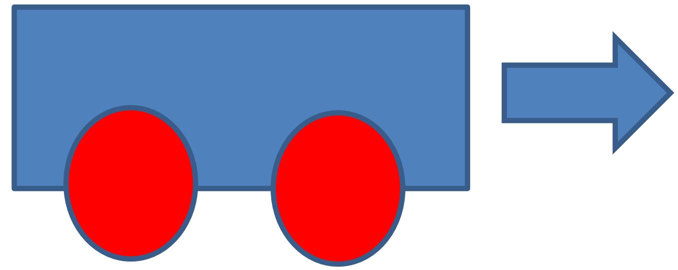
<u>Flux</u> dominant for most scales

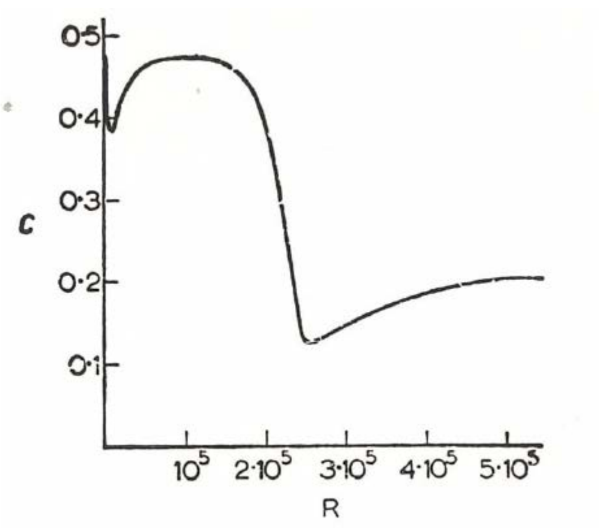
Why broad range scales?
What motivates cascade concept?

A) Planes, trains, automobiles…

<u>DRAG</u>

- Recall: $F_d \sim c_D \rho A V^2$
- $C_D = C_D(Re)$ → drag coefficient





- The Point:

  - Energy dissipation is finite, and due to viscosity, yet does not depend explicitly on viscosity ➔ $\boxed{\text{ANOMALY}}$

  - ‘Irreversibility persists as symmetry breaking factors vanish’

i.e. $\frac{dE}{dt} \sim F_d\, V \sim C_D \rho A V^3$

$\frac{dE}{dt} \sim \frac{V^3}{l_0} \equiv \epsilon \rightarrow$ dissipation rate $\quad l_0 \rightarrow$ macro length scale

- Where does the energy go?

  Steady state $\nu\langle(\nabla\vec{v})^2\rangle = \langle\vec{f}\cdot\vec{v}\rangle = \epsilon$

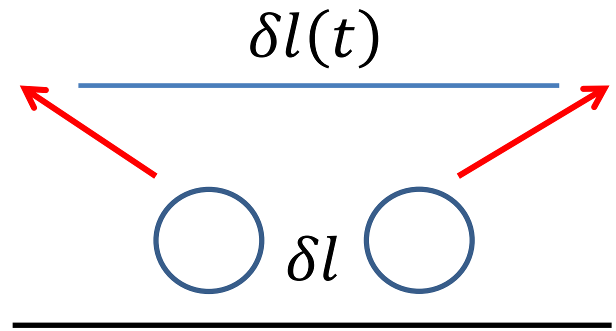
- So $\epsilon = \nu\langle(\nabla \mathrm{v})^2\rangle$ ← independent of ν

- $(\nabla v)_{rms} \sim \frac{1}{\nu^{1/2}}$ ➔ suggests → singular velocity gradients (small scale)

∴

- Flat C_D in Re → turbulence must access small scales as $Re \to \infty$
- Obviously consistent with broad spectrum, via nonlinear coupling

B) … and balloons



- Study of 'test particles' in turbulence:
- Anecdotal:

  Titus Lucretius Caro: 99-55 BC

  "De rerum Nature" cf. section V, line 500

- Systematic:

  L.F. Richardson: - probed atmospheric turbulence by study of balloon separation

  Noted: $\langle \delta l^2 \rangle \sim t^3$ → <u>super-diffusive</u>

  - not ~ t, ala' diffusion, noise

  - not exponential, ala' smooth chaotic flow

Upshot:

$\delta V(l) = \left(\left(\vec{v}(\vec{r} + \vec{l}) - \vec{v}(\vec{r}) \right) \cdot \frac{\vec{l}}{|\vec{l}|} \right)$ → structure function → velocity differential across scale

Then: $\delta V \sim l^{\alpha}$

so, $\frac{dl}{dt} \sim l^{\alpha}$ → growth of separation

→ $\langle l^2 \rangle \sim t^{\frac{2}{1-\alpha}} \sim t^3$

$\boxed{\rightarrow \alpha = \frac{1}{3}}$

so $\delta V(l) \sim l^{1/3}$, $\langle \delta l^2 \rangle \sim t^3$

→ Points:

- large eddys have more energy, so rate of separation increases with scale
- Relative separation is excellent diagnostic of flow dynamics

cf: tetrads: Siggia and Shraiman

Roughness:

N.B. turbulence is spatially "rough", i.e. $\delta V(l) \sim \epsilon^{1/3}\ l^{1/3}$

$$\lim_{l\to 0} \frac{V(\vec{r}+\vec{l}) - V(\vec{r})}{l} = \lim_{l\to 0} \frac{\delta V(l)}{l} = \epsilon^{1/3}\ /\ l^{2/3}$$

➔- strain rate increases on smaller scales

- turbulence develops progressively <u>rougher</u> structure on smaller scales

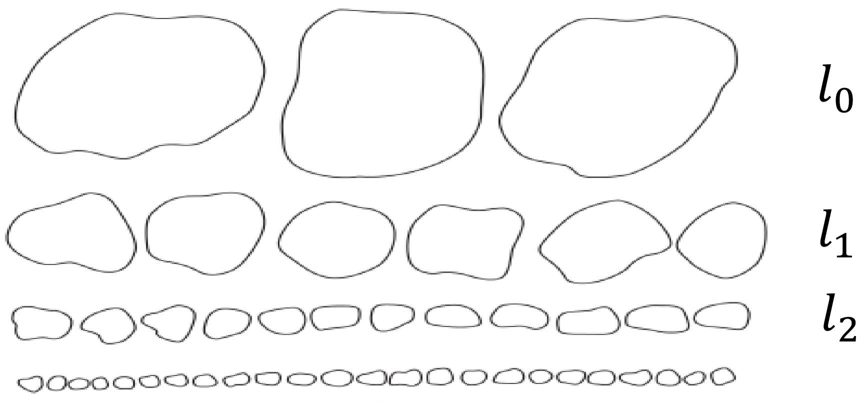
- Where are we?
  - turbulence develop singular gradients to maintain C_D indep. Re
  - turbulent flow structure exhibits
    - super-diffusive separation of test particles
    - power law scaling of $\delta V(l)$

➔

- Cascade model – K41

K41 Model (Phenomenological)

- Cascade → hierarchical fragmentation



- Broad range of scales, no gaps

- Described by structure function

- $\langle \delta V(l)^2 \rangle$, …. $\langle \delta V(l)^n \rangle$, ...

Related to energy distribution
←→ greatest interest

- $\langle \delta v(l)^2 \rangle \leftrightarrow$ energy,
of great interest

- higher moments
more challenging

- Input:
- 2/3 law (empirical)

  $S_2(l) \sim l^{2/3}$

- 4/5 law (Rigorous) - TBD

  $\langle \delta V(l)^3 \rangle = -\frac{4}{5}\epsilon l$

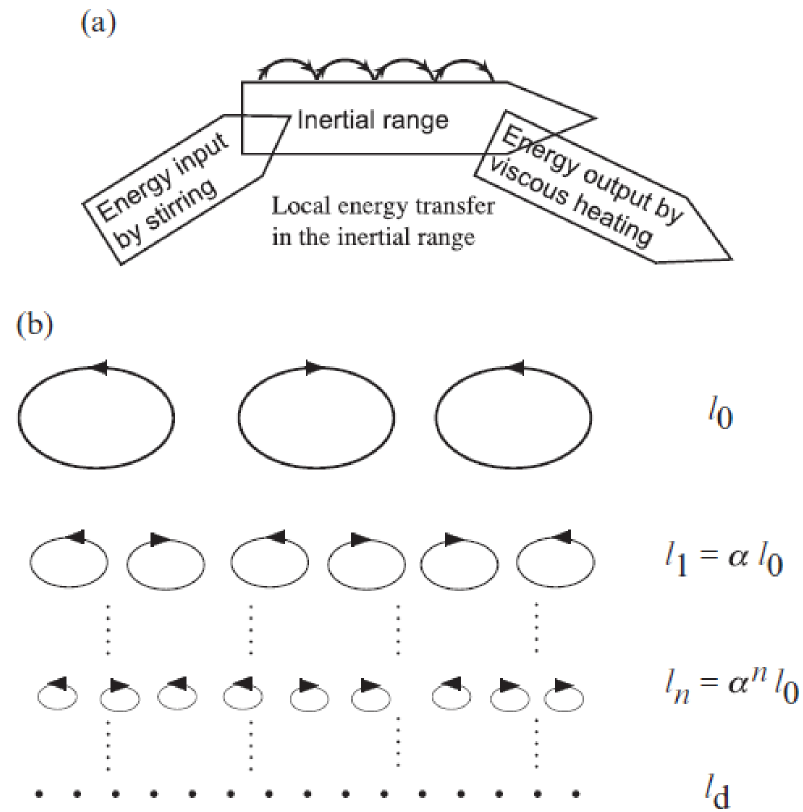


Fig. 2.12. Basic cartoon explanation of the Richardson–Kolmogorov cascade. Energy transfer in Fourier–space (a), and real scale (b)

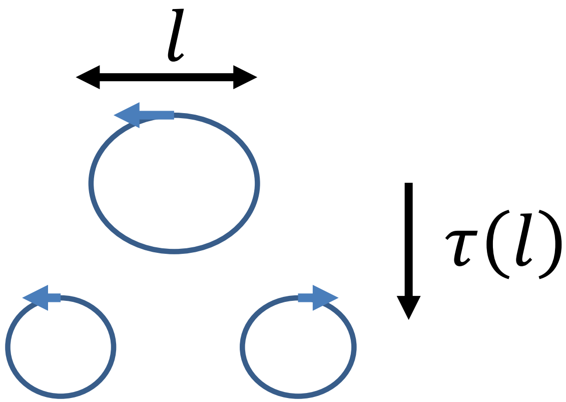
→ Ideas:

- <u>Flux</u> of energy in scale space from l_0 (input/integral scale) to l_d (dissipation) scale – set by ν
- Energy flux is <u>same</u> at all scales between l_0, l_d <-> self-similarity

And

- Energy dissipation – set as $\nu \to 0$ but not at $\nu = 0$
- * Asymmetry of breaking or stirring etc. lost in cascade: symmetry restoration
- N.B. intermittency <-> 'memory' of stirring, etc
- Ingredients / Players
  - Exciton → eddy (not a wave / eigenmode!)
  - l: scale parameter, eddy scale
  - $\delta V(l)$: velocity increment. Hereafter $V(l)$

- V_o: rms eddy fluctuation (large scale dominated)
- $\tau(l)$: → eddy transfer / life-time / turn-over rate
- → characteristic scale of transfer in cascade step



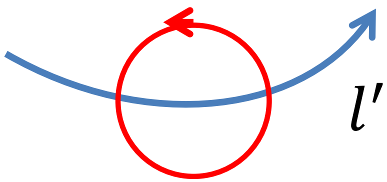
- Self-similarity → constant flow-thru rate $\epsilon = V(l)^2/\tau(l)$
- What is $\tau(l)$?? Consider…

The possibilities:

- Dimensionally, $\tau(l)$ is 'lifetime' of structure of scale l, time to distort out of existence

So

- $l' > l$



  - Larger scales advect eddy but don't distort it
  - Physics can't change under Galilean boost

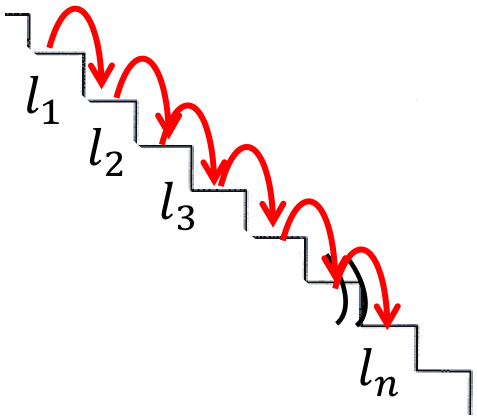
  cf: Rapid distortions, shearing

- $l' < l$
  - Irrelevant → insufficient energy

➔

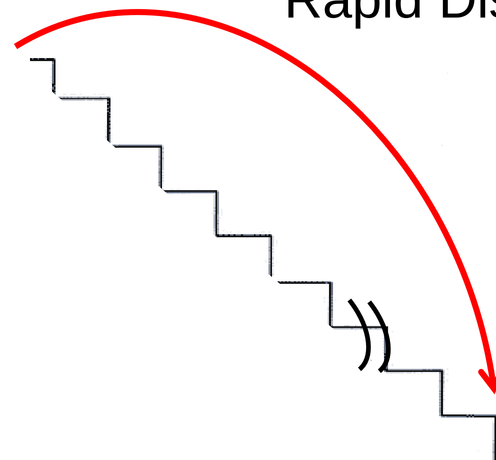
- $\tau(l) \sim l/V(l)$, set by $l' \sim l$

→ So



not

exception:
Rapid Distortion Theory



→ $\epsilon \sim V(l)^2 / \tau(l) \sim V(l)^3 / l$ → $V(l) \sim (\epsilon l)^{1/3}$; $1/\tau(l) \sim (\epsilon / l^2)^{1/3}$

→ $V(l)^2 \sim V_0^2 \, (l \, / \, l_0)^{2/3}$ (transfer rate increases as scale decreases)

And

→ $E(k) \sim \epsilon^{2/3} \, k^{-5/3}$ $\quad$ $E = \int dk E(k)$

→ Where does it end?

- Dissipation scale
  - cut-off at $1/\tau(l) \sim \nu/l^2$ i.e. $Re(l) \to 1$
  - $l_d \sim \nu^{3/4} / \epsilon^{1/4}$
- Degrees of freedom

$$\#DOFs \sim \left(\frac{l_0}{l_d}\right)^3 \sim Re^{9/4}$$

For $l_o \sim 1km$, $l_d \sim 1mm$ (PBL)

→ $N \sim 10^{18}$

→ Anything missing here?

- Dynamics!

i.e.
  - How is the energy transferred?
  - How are small scales generated?
  - Where have the N.S. equations gone?

  ...

- Enter vorticity!

- $\omega = \nabla \times \vec{v}\ ;\ \ \partial_t \vec{v} = \nabla \times \vec{v} \times \vec{\omega} + \nu \nabla^2 \vec{v}$

- $\Gamma = \int \oint \vec{v} \cdot d\vec{l} \sim const.$ to ν (Kelvin's theorem)

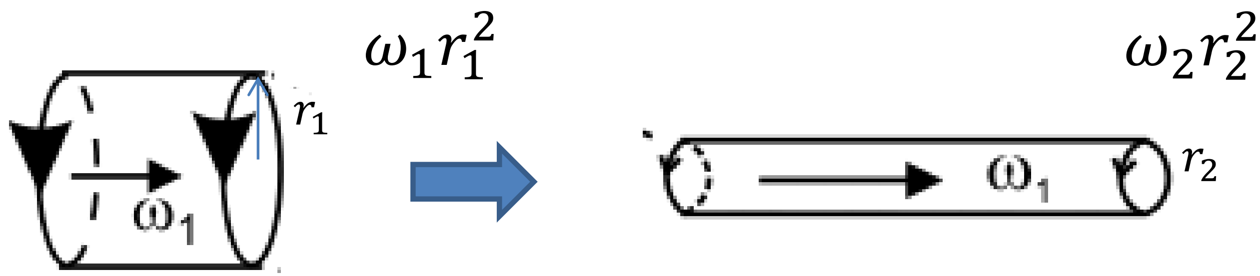
So

Vortex tube stretching

Strain tensor

- $\dfrac{\partial \vec{\omega}}{\partial t} + \vec{v} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{v} + \nu \nabla^2 \vec{\omega}$ $\qquad \vec{\omega} \cdot \overleftrightarrow{S}$

- Stretching:



  - Small scales generated ($\nabla \cdot \vec{v} = 0$)
  - Energy transferred to small scale

- Enstrophy $\Omega = \langle \omega^2 \rangle$

$$\frac{d\omega^2}{dt} = \vec{\omega} \cdot (\vec{\omega} \cdot \nabla \vec{v}) + \cdots \sim \omega^3 + \cdots$$

  - Enstrophy increases in 3D N-S turbulence
  - Growth is strongly nonlinear

- Enstrophy production underpins forward energy cascade

- Where are we?

“Big whorls have little whorls that feed on their velocity. And little whorls have lesser whorls. An so on to viscosity.” – L.F. Richardson, 1920

After: “So naturalists observe a flea has smaller fleas that on him prey; And these have smaller yet to bite ’em, And so proceed ad infinitum. Thus every poet, in his kind, Is bit by him that comes behind.” – Jonathan Swift, “On Poetry, a Rhapsody”, 1793

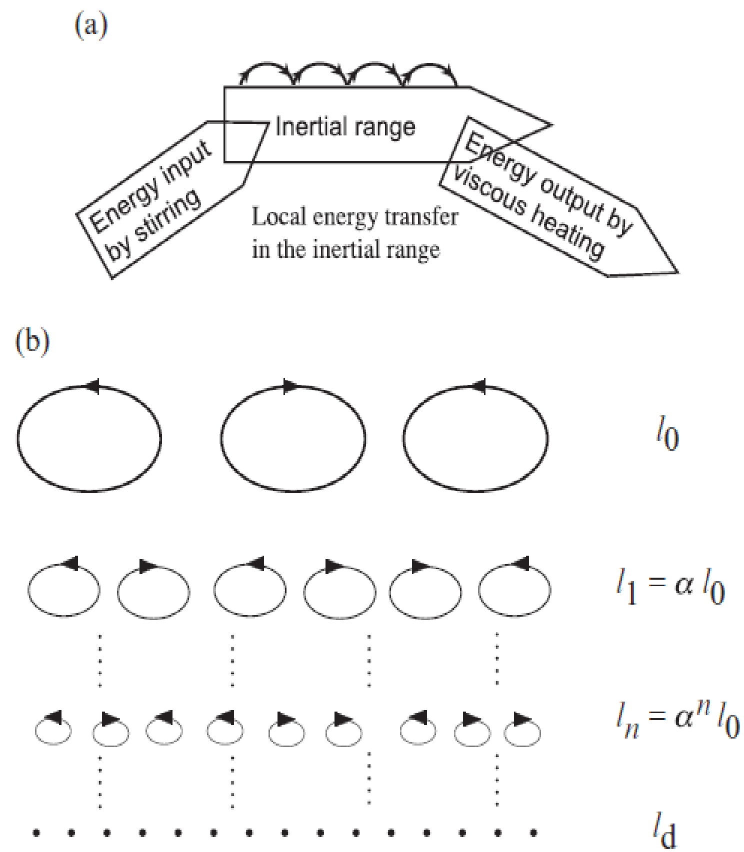


Fig. 2.12. Basic cartoon explanation of the Richardson–Kolmogorov cascade. Energy transfer in Fourier–space (a), and real scale (b)

The Theoretical Problem

- "We don't want to *think* anything, man. We want to *know*."

  – Marsellus Wallace, in "Pulp Fiction" (Quentin Tarantino)

- What do we know?

  – 4/5 Law (and not much else...)

$\langle V(l)^3 \rangle = -\frac{4}{5}\epsilon l$ → asymptotic for finite $l, \nu \to 0$

$S_2 = \langle \delta V(l)^2 \rangle$
$S_3 = \langle \delta V(l)^3 \rangle$

from: $\frac{\partial S_2}{\partial t} = -\frac{1}{3l^4}\frac{\partial}{\partial l}(l^4 S_3) - \frac{4}{3}\epsilon + \frac{2\nu}{l^4}\frac{\partial}{\partial l}\left(l^4 \frac{\partial S_2}{\partial l}\right)$

(Karman-Howarth)

flux in scale dissipation

- Stationarity, $\nu \to 0$

4/5 Law

- Asymptotically exact $\nu \to 0$, l finite
- Unique, rigorous result

- $\boxed{S_3(l) = -\frac{4}{5}\epsilon l}$
- Energy thru-put balance $\langle \delta V(l)^3 \rangle / l \leftrightarrow \epsilon$
- Notable:
  - Euler: $\partial_t v + v \cdot \nabla v + \nabla P/\rho = 0$; reversible; $t \to -t, v \to -v$
  - N-S: $\partial_t v + v \cdot \nabla v + \nabla P/\rho = \nu \nabla^2 v$; time reversal broken by viscosity
  - $S_3(l)$: $S_3(l) = -\frac{4}{5}\epsilon l$; reversibility breaking maintained as $\nu \to 0$

Anomaly

- Extensions:

  MHD: Pouquet, Politano

  2D: Celari, et. al. (inverse cascade, only)

What of so called ‘entropy cascade’ in Vlasov turbulence?

- N.B.: A little history; philosophy:
  - ‘Anomaly’ in turbulence → Kolmogorov, 1941
  - Anomaly in QFT → J. Schwinger, 1951 (regularization for vacuum polarization)
- Speaking of QFT, what of renormalized perturbation theory?
  - Renormalization gives some success to low order moments, identifies relevant scales
  - Useful in complex problems (i.e. plasmas) and problems where τ_{int} is not obvious
  - Rather few fundamental insights have emerged from R.P.T

    Caveat Emptor

Turbulence in Flat Land

- 2D systems → 1 dimension constrained

  i.e. Atmospheric <-> rotation Ω_0

  $V/L\,\Omega_{eff} < 1$

  Low Rossby number

  Magnetized plasma <-> $\overrightarrow{B_0},\ \Omega_c$

  Solar interior <-> stratification, ω_{B-V}

- Simple 2D fluid:

$$\frac{d\vec{\omega}}{dt} = \cancel{\vec{\omega}\cdot\nabla\vec{v}} + \nu\nabla^2\vec{\omega}$$

$$\vec{v} = \nabla\phi\times\hat{z}$$
$$\omega = -\nabla^2\phi$$

➔

$$\partial_t\nabla^2\phi + \nabla\phi\times\hat{z}\cdot\nabla\nabla^2\phi = \nu\nabla^2\nabla^2\phi + \tilde{s}$$

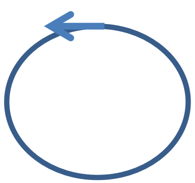
forcing scale variable

– ω constant along fluid trajectories, to ν

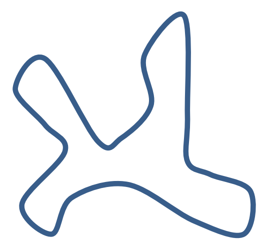
– $\omega = \nabla^2\phi$ akin conserved phase space density

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = C(f)$$

- The problem:
  - Enstrophy now conserved: $\vec{\omega} \cdot \nabla \vec{v} = 0$
  - Two inviscid invariants:
    - Enstrophy $\Omega = \langle (\nabla^2 \phi)^2 \rangle$
    - Energy $E = \langle (\nabla \phi)^2 \rangle$
  - Might ask: Where do these want to go, in scale?
  - Enstrophy:



+ turbulent flow ➔



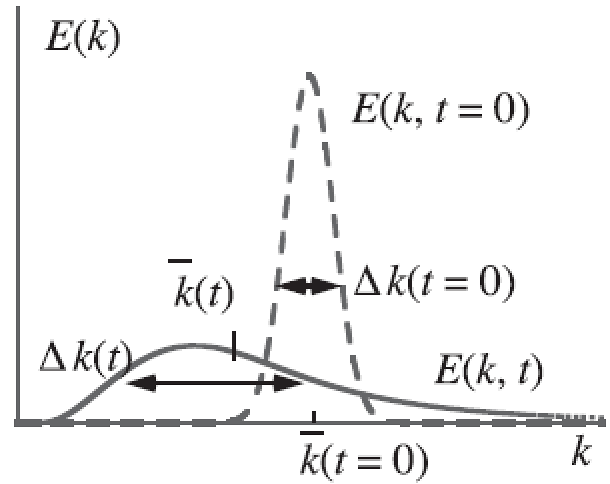
Isovorticity contour

Stretched contour, $\langle (\nabla \omega)^2 \rangle \uparrow$
→ Enstrophy to small scale

- Energy
  - Expect $(\Delta k)^2$ increases
  - What of centroid $\vec{k}$?

$$(\Delta k)^2 = \frac{1}{E}\int dk\left(k-\bar{k}\right)^2 E(k)$$

$$\bar{k} = \frac{1}{E}\int dk E(k)$$



But

$$(\Delta k)^2 = \frac{1}{E}\int dk\left(k^2 - 2k\bar{k} + \bar{k}^2\right)E(k) = \frac{1}{\Omega}\left(\Omega - \bar{k}^2\right)$$

$\partial_t(\Delta k)^2 > 0$ ➔ $\partial_t\bar{k} < 0$ Ω conserved!

➔ energy should head toward large scale

- Dilemma:
  - Energy seeks large scale
  - Enstrophy seeks small scale
  - How accommodate self-similar transfer – i.e. cascade – of both?

➔ Dual cascade (R.H. Kraichnan)

- Forward self-similar transfer of enstrophy

  → toward small scale dissipation

- Inverse transfer of energy

  → scale independent dissipation?

(Low k sink)

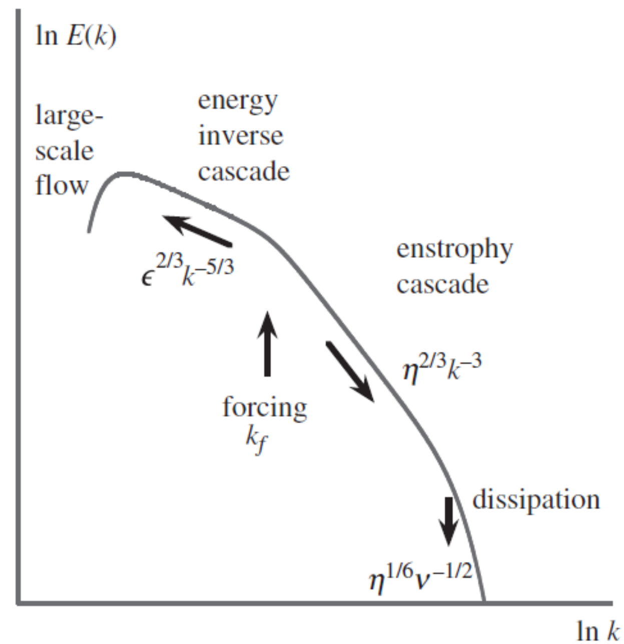


Fig. 2.17. Schematic of energy spectrum for dual cascade.

- Spectra
  - Enstrophy range:

    $E(l) \rightarrow kE(k)$

    $1/\tau(l) \rightarrow k[kE(k)]^{1/2}$

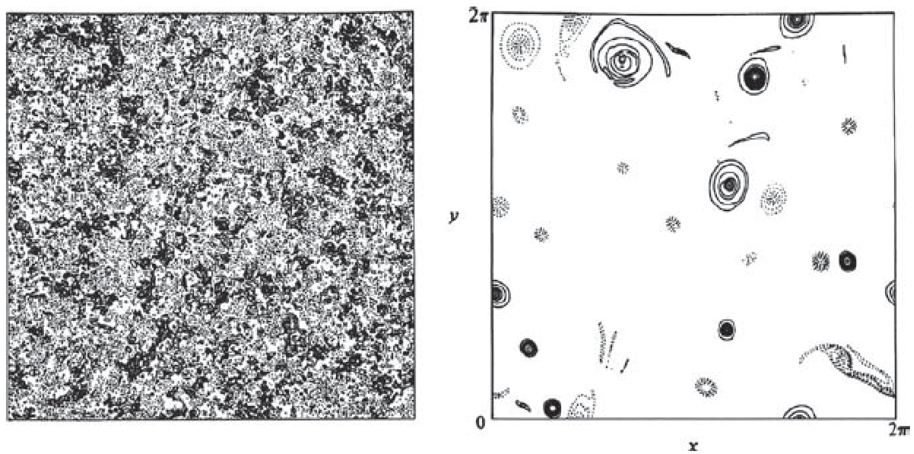
    → $E(k) = \eta^{2/3}\, k^{-3}$
  - Energy range: ala' K41; $E(k) = \epsilon^{2/3}\, k^{-5/3}$
- Pair dispersion:
  - Energy range: ala' Richardson
  - Enstrophy range: exponential divergence
- Scale independent dissipation critical to stationary state

→ Where do we stand now?

“Big whorls meet bigger whorls, And so it tends to go on. By merging they grow bigger yet, And bigger yet, and so on.”

- M. McIntyre, after L.F. Richardson

- Cautionary tale: coherent structures happen!



Decay experiment
→ Isolated coherent vortices appear in turbulent flow

McWilliams, '84 et. seq.
Herring and McWilliams '85

- Depending upon forcing, dynamics be cascade or coherent structure formation, or both:

- Need a non-statistical criterion, i.e. Okubo-Weiss

  $\rho = -\nabla^2\phi,\, S = \partial^2\phi/\,\partial x\partial y$ → local flow shear

  $\partial_t \nabla\rho = (s^2 - \rho^2)^{1/2}$; criterion for "coherence"

→ Gaussian curvature of stream function predicts stability

- <u>MHD turbulence</u> - A First Look
  - HUGE subject – includes small scale and mean field dynamo problems (c.f. Hughes lectures)
  - Here, focus on Alfvenic turbulence i.e. (Kraichnan-Iroshnikov-Goldreich-Sridhar …) → wave turbulence
    - Strong mean $\overrightarrow{B_0}$
    - $\delta B < B_0,\ \nabla \cdot \vec{v} = 0$
    - Shear-Alfven wave turbulence

$\vec{B}_0$

  - Best described by reduced MHD: (Ohm's Law, $\nabla \cdot J = 0$)

$$\frac{\partial A_{\parallel}}{\partial t} + \nabla_{\perp}\phi \times \hat{z} \cdot \nabla_{\perp} A_{\parallel} = B_0 \partial_z \phi + \eta \nabla^2 A_{\parallel}$$

$$\frac{\partial}{\partial t}\nabla^2\phi + \nabla_{\perp}\phi \times \hat{z} \cdot \nabla_{\perp}\nabla^2\phi^2 = B_0\partial_z\nabla^2 A_{\parallel} + \nabla_{\perp}A_{\parallel} \times \hat{z} \cdot \nabla_{\perp}\nabla^2 A_{\parallel} + \nu\nabla^2\nabla^2\phi + \tilde{S}$$

- Observations:
  - All nonlinear scattering is perpendicular
  - Contrast N-S, eddys with $\omega = 0$

    Now: Alfven waves: $\omega^2 = k_\parallel^2 V_A^2$
  - If uni-directional wave population:

    i.e. $A = f(z - V_A t) + \cancel{g(z + V_A t)}$

    then f is exact solution of MHD

  ➔ Need counter-propagating populations to manifest nonlinear interaction

  - See also resonance conditions
$$\omega_1 + \omega_2 = \omega_3$$
$$k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3}$$

- For Alfven wave cascade:

$$\epsilon = T(k \rightarrow k + \Delta k)E(k) \rightarrow E(k)/\tau(k)$$

transition rate

- Recall Fermi Golden Rule:

$$T_{i;j} \sim \frac{2\pi}{h} |< i|H_{int}|j >|^2 \, \delta\left(E_j - E_i - h\omega\right)$$

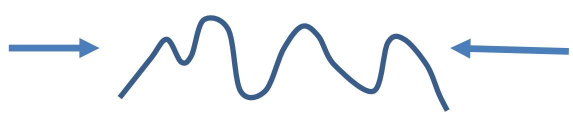
$$\rightarrow T \sim \frac{V(l_d)^2}{l^2} \, \tau_{int}\,(l_\perp)$$

$V(l_\perp)^2 \rightarrow$ scatter energy

$1/l^2 \rightarrow (cc)^2$

- $\tau_{int}(l) = 1/(\Delta k_\parallel)V_A$

$\rightarrow$ Alfvenic transit time ($\Delta k_\parallel \sim k_\parallel$)

Packet passage 

Enter the <u>Kubo number</u>

$$\frac{l_{\parallel ac}}{\Delta_\perp}\frac{\delta B}{B_0} \sim \left(\frac{V_A \delta B/B}{l_\perp}\right) \left|\Delta k_\parallel V_A\right|$$

- Basically: $B \cdot \nabla \rightarrow B_0 \partial_z + \tilde{B} \cdot \nabla_\perp$

  → relative size $\begin{cases} \text{Linear } B_0 \partial_z \\ \text{Nonlinear } \tilde{B} \cdot \nabla_\perp \end{cases}$

i.e. K < 1 → weak scattering, diffusion process

K > 1 → strong scattering, ~ de-magnetization ~ percolation

K = 1 → (critical) balance

Why Kubo?

- But… "It ain't over till its over"

  - Eastern (division) philosopher

- As $l_\perp$ drops, $V(l_\perp)/\, l_\perp \rightarrow (\Delta k_\parallel) V_A$

➔ $\tau_\perp \rightarrow \tau_\parallel$ $\qquad Ku \rightarrow 1$

- Critically balanced cascade, $Ku \sim 1$

  i.e. $\frac{V(l_\perp)}{l_\perp} \sim V_A \frac{\delta B(l_\perp)}{B_0} \sim (\Delta k_\parallel) V_A$, unavoidable at small scale

  – Statement that transfer sets $K \approx 1$

  – Attributed to G.-S. '95 but:

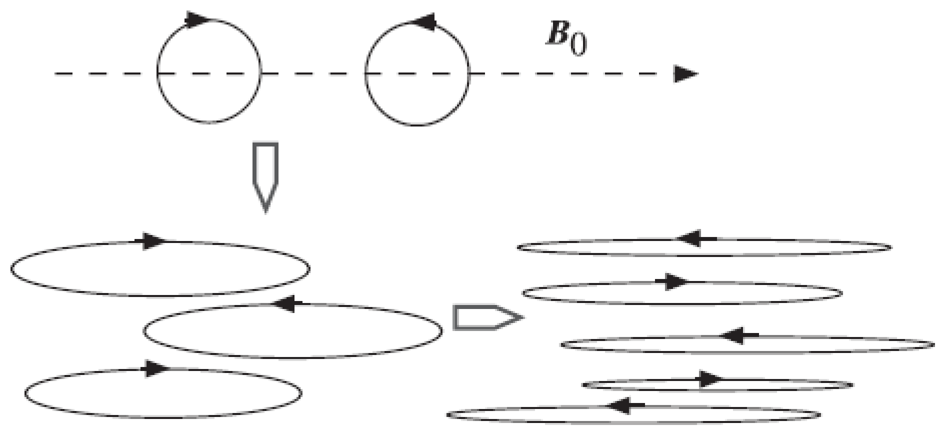
  "the natural state of EM turbulence is K ~ 1"

  - Kadomtsev and Pogutse '78

$$\boxed{k_\parallel = k_\parallel(l_\perp)}$$

defines anisotropy

- If now $\frac{1}{\tau_{int}(l_\perp)} \sim \frac{V(l_\perp)}{l_\perp}$
  - Recover K41 scaling in MHDT, $F(k_\perp) \sim \epsilon^{\frac{2}{3}} k_\perp^{-\frac{5}{3}}$
  - "Great Power Law in the Sky"
- Eddy structure:



$$k_\| V_A \sim \frac{V(l_\perp)}{l_\perp} \Rightarrow k_\| \sim k_\perp^{\frac{2}{3}} \epsilon^{\frac{1}{3}} / V_A$$ ➔ anisotropy increases as $l_\perp \downarrow$

- Many variants, extensions, comments, "we did it too's"…

➔ Fate of Energy?

- End point is dissipation
- What is dissipative structure?
  - Dimension < 3 → fractal and multi-fractal intermittency models
  - Structure:
    - Vortex sheet
    - Current sheet

    → Stability → micro-tearing, etc.
  - Energy leak to kinetic scales?
    - Electron vs ion heating
    - Particle acceleration (2nd order Fermi)

Conclusion

- This lecture is not even the "end of the beginning"
- A few major omissions:
  - pipe flow turbulence – Prandtl law of the wall
  - spatial structures, mixing, spreading

    P.D.
  - general theory of wave turbulence - Qiu, P.D.
  - MHDT + small scale dynamo - Hughes
  - kinetic/Vlasov turbulence – Sarazin, Qiu, Dif-Pradalier
  - Langmuir collapse … - Kosuga